THE CHINESE UNIVERSITY OF HONG KONG
Department of Mathematics
MATH3070 (Second Term, 2017–2018)
Introduction to Topology
Exercise 7 Product

Remarks

Many of these exercises are adopted from the textbooks (Davis or Munkres). You are suggested to work more from the textbooks or other relevant books.

1. Show that the relative topology (induced topology) is "transitive" in some sense. That is for $A \subset B \subset (X, \mathfrak{T})$, the topology of A induced indirectly from B is the same as the one directly induced from X.

2. Let $A \subset (X, \mathfrak{T})$ be given the induced topology $\mathfrak{T}|_A$ and $B \subset A$. Guess and prove the relation between $\mathrm{Int}_A(B)$ and $\mathrm{Int}_X(B)$ which are the interior wrt to $\mathfrak{T}|_A$ and $\mathfrak{T}$. Do the similar thing for closures.

3. Let $A \subset (X, \mathfrak{T})$ be given a topology $\mathfrak{T}_A$. Formulate a condition for $\mathfrak{T}_A$ being the induced topology in terms of the inclusion mapping $\iota\colon\ A \hookrightarrow X$.

4. Let $Y \subset (X, \mathfrak{T})$ be a closed set which is given the induced topology. If $A \subset Y$ is closed in $(Y, \mathfrak{T}|_Y)$, show that A is also closed in $(X, \mathfrak{T})$.

5. Let $X \times X$ be given the product topology of $(X, \mathfrak{T})$. Show that $D = \{\, (x, x) : x \in X \,\}$ as a subspace of $X \times X$ is homeomorphic to X.

6. Let Y be a subspace of $(X, \mathfrak{T})$, i.e., with the induced topology and $f\colon\ X \to Z$ be continuous. Is the restriction $f|_Y\colon\ Y \to Z$ continuous?

7. Show that $(X \times Y) \times Z$ is homeomorphic to $X \times (Y \times Z)$ wrt product topologies.

8. Let $X_1 \times X_2$ be given the product topology. Show that the mappings $\pi_j\colon\ X_1 \times X_2 \to X_j$, $j = 1, 2$, are open and continuous.

   Moreover, let $\mathfrak{T}^*$ be a topology on $X_1 \times X_2$ such that both mappings

   $$\pi_j\colon\ (X_1 \times X_2, \mathfrak{T}^*) \to (X_j, \mathfrak{T}_j), \quad j = 1, 2,$$

   are continuous. What is the relation between $\mathfrak{T}^*$ and the product topology?

9. Given any topological space Y and product space $X_1 \times X_2$, a mapping $f\colon\ Y \to X_1 \times X_2$ is continuous if and only if $\pi_j \circ f$, $j = 1, 2$, are continuous.

   If $\mathfrak{T}^*$ is a topology on $X_1 \times X_2$ with the same property, then $\mathfrak{T}^*$ is the product topology.

10. Let (X_n, d_n), $n \in \mathbb{N}$, be a countable family of metric spaces; $X = \prod_{n=1}^{\infty} X_n$ be the product space of the metric topologies induced by d_n. Define a metric d on X in this way, for $x = (x_n), y = (y_n) \in X$,

$$d(x, y) = \sum_{n=1}^{\infty} \frac{1}{2^n} \frac{d_n(x_n, y_n)}{1 + d_n(x_n, y_n)} .$$

Show that d is a metric on X and the topology it induces is exactly the product topology.

11. Let $[0, 1]$ and $(0, 1]$ be intervals having the induced topology from the standard $\mathbb{R}$. Prove that the product spaces $[0, 1] \times (0, 1]$ and $(0, 1] \times (0, 1]$ are homeomorphic.

12. Let $\mathbb{R}$ be given the standard topology; and $\mathbb{R}_{\ell\ell}$ be the one with lower-limit topology. What is the induced topology on the diagonal $\{ (x, x) : \ x \in \mathbb{R} \}$ from $\mathbb{R} \times \mathbb{R}_{\ell\ell}$?

13. Let $X = R^{\mathbb{N}}$ be given the product topology of standard $\mathbb{R}$. Denote $0 \in X$ the constant zero function and a sequence of functions $x_n \in X$ is defined by $x_n(k) = 0$ for $k \leq n$ while $x_n(k) = 1$ for $k > n$. Show that $x_n \to 0$ in X.

14. Given topological spaces $(X_\alpha, \mathfrak{T}_\alpha)$ and let

$$\mathcal{B}_{\text{box}} = \left\{ \prod_\alpha U_\alpha : \ U_\alpha \in \mathfrak{T}_\alpha \right\} .$$

Show that $\mathcal{B}_{\text{box}}$ also defines a topology $\mathfrak{T}_{\text{box}}$ for $\prod_\alpha X_\alpha$. It is called the box product.